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Sharp estimates for semi-stable radial solutions of semilinear elliptic equations

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Abstract

This paper is devoted to the study of semi-stable radial solutions $u \in H^1(B_1)$ of $-\Delta u = g(u)$ in $B_1 \setminus \{0\}$, where $g \in C^1(\mathbb{R})$ is a general nonlinearity and B_1 is the unit ball of $\mathbb{R}^N$. We establish sharp pointwise estimates for such solutions. As an application of these results, we obtain optimal pointwise estimates for the extremal solution and its derivatives (up to order three) of the semilinear elliptic equation $-\Delta u = \lambda f(u)$, posed in B_1, with Dirichlet data $u|_{\partial B_1} = 0$, where f is a continuous, positive, nondecreasing and convex function on $[0, \infty)$ such that $f(s)/s \to \infty$ as $s \to \infty$. In addition, we provide, for $N \geqslant 10$, a large family of semi-stable radially decreasing unbounded $H^1(B_1)$ solutions.

Keywords: Sharp estimates; Stability; Elliptic equations

1. Introduction and main results

This paper deals with the semi-stability of radial solutions $u \in H^1(B_1)$ of

$$-\Delta u = g(u) \quad \text{in } B_1 \setminus \{0\}, \tag{1.1}$$

where B_1 is the unit ball of $\mathbb{R}^N$, and $g \in C^1(\mathbb{R})$ is a general nonlinearity.

A radial solution $u \in H^1(B_1)$ of (1.1) is called semi-stable if

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$$\int_{B_1} \left(|\nabla v|^2 - g'(u)v^2\right) dx \geqslant 0$$

for every $v \in C^\infty(B_1)$ with compact support in $B_1 \setminus \{0\}$.

As an application of some general results obtained in this paper for this class of solutions (for arbitrary $g \in C^1(\mathbb{R})$), we will establish sharp pointwise estimates related to the following semilinear elliptic equation, which has been extensively studied.

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain, $N \geqslant 1$, $\lambda \geqslant 0$ is a real parameter, and the nonlinearity $f : [0, \infty) \to \mathbb{R}$ satisfies

$$f \text{ is } C^1, \text{ nondecreasing and convex, } f(0) > 0, \text{ and } \lim_{u \to +\infty} \frac{f(u)}{u} = +\infty. \tag{1.2}$$

It is well known that there exists a finite positive extremal parameter λ^* such that (P_λ) has a minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 \leqslant \lambda < \lambda^*$, while no solution exists, even in the weak sense, for $\lambda > \lambda^*$. The set $\{u_\lambda\colon\ 0 \leqslant \lambda < \lambda^*\}$ forms a branch of classical solutions increasing in λ. Its increasing pointwise limit $u^*(x) := \lim_{\lambda \uparrow \lambda^*} u_\lambda(x)$ is a weak solution of (P_λ) for $\lambda = \lambda^*$, which is called the extremal solution of (P_λ) (see [1,2]).

The regularity and properties of the extremal solutions depend strongly on the dimension N, domain Ω and nonlinearity f. When $f(u) = e^u$, it is known that $u^* \in L^\infty(\Omega)$ if $N < 10$ (for every Ω) (see [7,10]), while $u^*(x) = -2\log|x|$ and $\lambda^* = 2(N-2)$ if $N \geqslant 10$ and $\Omega = B_1$ (see [9]). There is an analogous result for $f(u) = (1+u)^p$ with $p > 1$ (see [2]). Brezis and Vázquez [2] raised the question of determining the boundedness of u^*, depending on the dimension N, for general nonlinearities f satisfying (1.2). The best result is due to Nedev [11], who proved that $u^* \in L^\infty(\Omega)$ if $N \leqslant 3$, and Cabré [3], who has proved recently that $u^* \in L^\infty(\Omega)$ if $N = 4$ and Ω is convex. Cabré and Capella [5] have proved that $u^* \in L^\infty(\Omega)$ if $N \leqslant 9$ and $\Omega = B_1$ (similar results for the p-Laplacian operator are contained in [6]). Another interesting question is whether the extremal solution lies in the energy class. Nedev [11] proved that $u^* \in H_0^1(\Omega)$ if $N \leqslant 5$ (for every Ω). Brezis and Vázquez [2] proved that a sufficient condition to have $u^* \in H_0^1(\Omega)$ is that $\liminf_{u\to\infty} uf'(u)/f(u) > 1$ (for every Ω and $N \geqslant 1$). On the other hand, it is an open problem (see [2, Problem 5]) to know the behavior of $f'(u^*)$ near the singularities of u^*. Is it always like $C/|x|^2$?

If $\Omega = B_1$, it is easily seen by the Gidas–Ni–Nirenberg symmetry result that u_λ is radially decreasing for $0 < \lambda < \lambda^*$. Hence, its limit u^* is also radially decreasing. In this situation, Cabré and Capella [5] have proved the following result:

Theorem 1.1. *(See [5].) Assume that $\Omega = B_1$, $N \geqslant 2$, and that f satisfies* (1.2). *Let u^* be the extremal solution of* (P_λ). *We have that*

(i) *if $N < 10$, then $u^* \in L^\infty(B_1)$,*
(ii) *if $N = 10$, then $u^*(x) \leqslant C|\log|x||$ in B_1 for some constant C,*
(iii) *if $N > 10$, then $u^*(x) \leqslant C|x|^{-N/2+\sqrt{N-1}+2}\sqrt{|\log|x||}$ in B_1 for some constant C,*

(iv) *if* $N \geqslant 10$ *and* $k \in \{1,2,3\}$, *then* $|\partial^k u^*(x)| \leqslant C|x|^{-N/2+\sqrt{N-1}+2-k}\sqrt{|\log|x||}$ *in* B_1 *for some constant* C.

Among other results, in this paper we establish sharp pointwise estimates for u^* and its derivatives (up to order three) in the radial case. We improve the above theorem, answering affirmatively to an open question raised in [5], about the removal of the factor $\sqrt{|\log|x||}$.

By abuse of notation, we write $u(r)$ instead of $u(x)$, where $r = |x|$ and $x \in \mathbb{R}^N$. We denote by u_r the radial derivative of a radial function u.

Theorem 1.2. *Assume that* $\Omega = B_1$, $N \geqslant 1$, *and that* f *satisfies* (1.2). *Let* u^* *be the extremal solution of* (P_λ). *We have that*

(i) *If* $N < 10$, *then* $u^*(r) \leqslant C(1-r)$, $\forall r \in [0,1]$,
(ii) *If* $N = 10$, *then* $u^*(r) \leqslant C|\log r|$, $\forall r \in (0,1]$,
(iii) *If* $N > 10$, *then* $u^*(r) \leqslant C(r^{-N/2+\sqrt{N-1}+2} - 1)$, $\forall r \in (0,1]$,
(iv) *If* $N \geqslant 10$, *then* $|\partial_r^{(k)} u^*(r)| \leqslant Cr^{-N/2+\sqrt{N-1}+2-k}$, $\forall r \in (0,1]$, $\forall k \in \{1,2,3\}$,

where $C = C_N \min_{t\in[1/2,1]} |u_r^*(t)|$, *and* C_N *is a constant depending only on* N.

Remark 1.3. It is immediate that if we replace the function f by $\tilde{f} := f(\cdot/M)$, with $M > 0$, then the extremal solution $\tilde{u}^*$ associated to $\tilde{f}$ is $\tilde{u}^* = Mu^*$. Hence the constant C in Theorem 1.2 must depend homogeneously on u^*. In fact, this linear coefficient is very small since, for instance, we have

$$\min_{t\in[1/2,1]} \left|u_r^*(t)\right| \leqslant 4\left(u^*(1/2) - u^*(3/4)\right) \leqslant 4u^*(1/2) \leqslant \frac{4}{\text{measure}(B_{1/2})} \left\|u^*\right\|_{L^1(B_{1/2})}.$$

Remark 1.4. In [2] it is proved that if

$$N > 10 \quad \text{and} \quad p \geqslant p_N := \frac{N - 2\sqrt{N-1}}{N - 2\sqrt{N-1} - 4},$$

then the extremal solution for $f(u) = (1+u)^p$ and $\Omega = B_1$ is given by $u^*(r) = r^{-2/(p-1)} - 1$. In particular, if $N > 10$ and $p = p_N$ (called the Joseph–Lundgren exponent), then $u^*(r) = r^{-N/2+\sqrt{N-1}+2} - 1$. Hence the pointwise estimates of Theorem 1.2 for u^* and its derivatives (up to order three) are optimal if $N > 10$. The optimality of the theorem for $N = 10$ follows immediately by considering $f(u) = e^u$. As mentioned before, it is obtained in this case that $u^*(r) = 2|\log r|$.

Remark 1.5. In fact, the convexity of f is not necessary to obtain our main results. Specifically, if we assume $f \in C^1$, nondecreasing, $f(0) > 0$ and $\lim_{u\to+\infty} f(u)/u = +\infty$, then it can be proved (see [5, Proposition 5.1]) that there exits a finite positive extremal parameter λ^* such that (P_λ) has a minimal classical solution $u_\lambda \in C^2(\overline{\Omega})$ if $0 \leqslant \lambda < \lambda^*$, while no solution exists, even in the weak sense, for $\lambda > \lambda^*$. The set $\{u_\lambda\colon 0 \leqslant \lambda < \lambda^*\}$ of classical solutions is increasing in λ and its pointwise limit $u^*(x) := \lim_{\lambda\uparrow\lambda^*} u_\lambda(x)$ is a semi-stable weak solution of (P_λ) for $\lambda = \lambda^*$. Note that the family of minimal solutions $\{u_\lambda\}$ may not be continuous as a function of λ, as in the case of f convex. Under these hypotheses of f it is possible to obtain the results (with the only exception of the case $N \geqslant 10$ and $k = 3$ of item (iv)) of Theorems 1.1 and 1.2.

As we have mentioned, the proof of Theorem 1.2 is based on general properties of semi-stable radial solutions. Note that the minimality of u_λ implies its semi-stability. Clearly, we can pass to the limit and obtain that u^* is also radial and semi-stable. In addition, in [5] it is proved that $u^* \in H^3(B_1)$ for every dimension N. In particular, $u^* \in H^1(B_1)$.

Recalling the definition of the semi-stability at the beginning of the paper, we observe that a radial solution $u \in H^1(B_1)$ of (1.1) is bounded away from the origin. Hence, using standard regularity results, we obtain $u \in C^2(B_1 \setminus \{0\})$, and the definition of semi-stability makes sense.

If u is a bounded radial solution of (1.1), then $u \in C^2(\overline{B_1})$ and the semi-stability of u means that the first eigenvalue of the linearized problem $-\Delta - g'(u)$ in B_1 is nonnegative.

Note that the expression which defines the semi-stability is nothing but the second variation of the energy functional associated to (1.1) in a domain $\Omega \subset \mathbb{R}^N$ (with $\overline{\Omega} \subset B_1 \setminus \{0\}$): $E_\Omega(u) = \int_\Omega (|\nabla u|^2/2 - G(u))\,dx$, where $G' = g$. Thus, if $u \in C^2(B_1 \setminus \{0\})$ is a local minimizer of E_Ω for every smooth domain $\Omega \subset \mathbb{R}^N$ (with $\overline{\Omega} \subset B_1 \setminus \{0\}$) (i.e., a minimizer under every small enough $C^1(\Omega)$ perturbation vanishing on $\partial\Omega$), then u is a semi-stable solution of (1.1). Other general situations include stable solutions: minimal solutions, extremal solutions or absolute minimizers between a subsolution and a supersolution (see [5, Rem. 1.11] for more details).

Our main results about semi-stable radial solutions are the following.

Theorem 1.6. *Let $N \geqslant 2$, $g \in C^1(\mathbb{R})$, and $u \in H^1(B_1)$ be a semi-stable radial solution of* (1.1). *Then there exists a constant M_N depending only on N such that*:

(i) *If $N < 10$, then* $\|u\|_{L^\infty(B_1)} \leqslant M_N \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})}$.
(ii) *If $N = 10$, then* $|u(r)| \leqslant M_{10} \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})} (|\log r| + 1)$, $\forall r \in (0, 1]$.
(iii) *If $N > 10$, then* $|u(r)| \leqslant M_N \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})} r^{-N/2+\sqrt{N-1}+2}$, $\forall r \in (0, 1]$.

Theorem 1.7. *Let $N \geqslant 2$, $g \in C^1(\mathbb{R})$, and $u \in H^1(B_1)$ be a semi-stable radially decreasing solution of* (1.1). *Then there exists a constant M'_N depending only on N such that*:

(i) *If $g \geqslant 0$, then*

$$\left|u_r(r)\right| \leqslant M'_N \|\nabla u\|_{L^2(B_1 \setminus B_{1/2})} r^{-N/2+\sqrt{N-1}+1}, \quad \forall r \in (0, 1/2].$$

(ii) *If $g \geqslant 0$ is nondecreasing, then*

$$\left|u_{rr}(r)\right| \leqslant M'_N \|\nabla u\|_{L^2(B_1 \setminus B_{1/2})} r^{-N/2+\sqrt{N-1}}, \quad \forall r \in (0, 1/2].$$

(iii) *If $g \geqslant 0$ is nondecreasing and convex, then*

$$\left|u_{rrr}(r)\right| \leqslant M'_N \|\nabla u\|_{L^2(B_1 \setminus B_{1/2})} r^{-N/2+\sqrt{N-1}-1}, \quad \forall r \in (0, 1/2].$$

Remark 1.8. We emphasize that the estimates obtained in Theorems 1.6 and 1.7 are in terms of the H^1 norm of the annulus $B_1 \setminus \overline{B_{1/2}}$, while u is required to belong to $H^1(B_1)$. In fact, this requirement is essential to obtain our results, since we can always find radial weak solutions of (1.1) (not in the Sobolev space of the unit ball), for which the statements of Theorems 1.6 and 1.7 fail to satisfy (see [2,5]).

Remark 1.9. In [5, Rem. 1.9] it is raised the question whether the estimates of Theorem 1.7 hold for general nonlinearities g, without the assumptions on the nonnegativeness of g, g' and/or g''. In this paper we answer negatively to this question. In fact, without assumptions on the sign of g, g' or g'' it is not possible to obtain any pointwise estimate for $|u_r|$, $|u_{rr}|$ or $|u_{rrr}|$ (see Corollaries 3.4, 3.6 and 3.9).

To prove the main results of the paper we will use Lemma 2.1, which, roughly speaking, says that there are some restrictions on the growth of the derivative of a radial semi-stable solution of (1.1) around the origin. In the proof of this lemma, we will make use of [5, Lem. 2.1], which was inspired by the proof of Simons theorem on the nonexistence of singular minimal cones in $\mathbb{R}^N$ for $N \leqslant 7$ (see [8, Th. 10.10] and [5, Rem. 2.2] for more details). Similar methods are used in [4,12] to study the stability or instability of radial solutions in all space $\mathbb{R}^N$.

The paper is organized as follows. In Section 2 we prove Theorems 1.2, 1.6 and 1.7. Section 3 provides, for $N \geqslant 10$, a large family of semi-stable radially decreasing unbounded $H^1(B_1)$ solutions of problems of the type (1.1). Taking solutions of this family, we will show the impossibility of obtaining pointwise estimates for $|u_r|$, $|u_{rr}|$ or $|u_{rrr}|$ if no further assumptions on the sign of g, g' or g'' are imposed.

2. Proof of the main results

Lemma 2.1. *Let $N \geqslant 2$, $g \in C^1(\mathbb{R})$, and $u \in H^1(B_1)$ be a semi-stable radial solution of* (1.1). *Then there exists a constant K_N depending only on N such that*:

$$\int_0^r t^{N-1} u_r(t)^2\, dt \leqslant K_N \|\nabla u\|^2_{L^2(B_1 \setminus B_{1/2})} r^{2\sqrt{N-1}+2} \quad \forall r \in [0,1]. \tag{2.1}$$

Proof. Let us use [5, Lem. 2.1] (see also the proof of [5, Lem. 2.3]) to assure that

$$(N-1)\int_{B_1} u_r^2 \eta^2\, dx \leqslant \int_{B_1} u_r^2 \left|\nabla(r\eta)\right|^2 dx,$$

for every $\eta \in (H^1 \cap L^\infty)(B_1)$ with compact support in B_1 and such that $|\nabla(r\eta)| \in L^\infty(B_1)$. Applying this inequality to a radial function $\eta(|x|)$ we obtain

$$(N-1)\int_0^1 u_r(t)^2 \eta(t)^2 t^{N-1}\, dt \leqslant \int_0^1 u_r(t)^2 \big(t\eta(t)\big)'^2 t^{N-1}\, dt. \tag{2.2}$$

We now fix $r \in (0, 1/2)$ and consider the function

$$\eta(t)=\begin{cases} r^{-\sqrt{N-1}-1} & \text{if } 0\leqslant t\leqslant r,\\ t^{-\sqrt{N-1}-1} & \text{if } r<t\leqslant 1/2,\\ 2^{\sqrt{N-1}+2}(1-t) & \text{if } 1/2<t\leqslant 1.\end{cases}$$

Since $(N-1)\eta(t)^2=(t\eta(t))'^2$ for $r<t<1/2$, inequality (2.2) shows that

$$\begin{aligned}
&(N-2)r^{-2\sqrt{N-1}-2}\int_0^r u_r(t)^2t^{N-1}\,dt\\
&\quad=\int_0^r \big((N-1)\eta(t)^2-\big(t\eta(t)\big)'^2\big)u_r(t)^2t^{N-1}\,dt\\
&\quad\leqslant-\int_{1/2}^1 \big((N-1)\eta(t)^2-\big(t\eta(t)\big)'^2\big)u_r(t)^2t^{N-1}\,dt\leqslant\alpha_N\int_{1/2}^1 u_r(t)^2t^{N-1}\,dt,
\end{aligned}$$

where the constant $\alpha_N=\max_{1/2\leqslant t\leqslant 1}-((N-1)\eta(t)^2-(t\eta(t))'^2)$ depends only on N. This establishes (2.1) for $r\in[0,1/2]$, if $N>2$.

If $r\in(1/2,1]$ and $N>2$ then, applying the above inequality for $r=1/2$, we obtain

$$\begin{aligned}
\int_0^r t^{N-1}u_r(t)^2\,dt&\leqslant\int_0^{1/2} t^{N-1}u_r(t)^2\,dt+\int_{1/2}^1 t^{N-1}u_r(t)^2\,dt\\
&\leqslant\left(\frac{\alpha_N}{N-2}\left(\frac{1}{2}\right)^{2\sqrt{N-1}+2}+1\right)\int_{1/2}^1 t^{N-1}u_r(t)^2\,dt\\
&\leqslant(2r)^{2\sqrt{N-1}+2}\left(\frac{\alpha_N}{N-2}\left(\frac{1}{2}\right)^{2\sqrt{N-1}+2}+1\right)\int_{1/2}^1 t^{N-1}u_r(t)^2\,dt
\end{aligned}$$

which is the desired conclusion with $K_N=\frac{1}{\omega_N}(\frac{\alpha_N}{N-2}+2^{2\sqrt{N-1}+2})$ (note that the constant obtained for $r\in(1/2,1]$ is greater than the one for $r\in[0,1/2]$).

Finally, if $N=2$, changing the definition of $\eta(t)$ in $[0,r]$ by $\eta(t)=1/(rt)$, if $r_0<t\leqslant r$; $\eta(t)=1/(rr_0)$, if $0\leqslant t\leqslant r_0$ (for arbitrary $r_0\in(0,r)$), we obtain

$$\frac{1}{r^2}\int_{r_0}^r\frac{u_r(t)^2}{t}\,dt\leqslant\alpha_2\int_{1/2}^1 u_r(t)^2t\,dt.$$

Letting $r_0\to 0$ and taking into account that $t/r^2\leqslant 1/t$ for $0<t\leqslant r$ yields (2.1) for $N=2$ and $r\in[0,1/2]$. If $r\in(1/2,1]$, we can apply similar arguments to the case $N>2$ to complete the proof. □

Proposition 2.2. *Let $N \geqslant 2$, $g \in C^1(\mathbb{R})$, and $u \in H^1(B_1)$ be a semi-stable radial solution of* (1.1). *Then there exists a constant K'_N depending only on N such that*:

$$\left|u(r) - u\left(\frac{r}{2}\right)\right| \leqslant K'_N \|\nabla u\|_{L^2(B_1 \setminus B_{1/2})} r^{-N/2+\sqrt{N-1}+2} \quad \forall r \in (0,1]. \tag{2.3}$$

Proof. Fix $r \in (0,1]$. Applying Cauchy–Schwarz and Lemma 2.1 we deduce

$$\begin{aligned}
\left|u(r) - u\left(\frac{r}{2}\right)\right| &\leqslant \int_{r/2}^{r} \left|u_r(t)\right| t^{\frac{N-1}{2}} \frac{1}{t^{\frac{N-1}{2}}}\, dt \\
&\leqslant \left(\int_{r/2}^{r} u_r(t)^2 t^{N-1}\, dt\right)^{1/2} \left(\int_{r/2}^{r} \frac{1}{t^{N-1}}\, dt\right)^{1/2} \\
&\leqslant K_N^{1/2} \|\nabla u\|_{L^2(B_1 \setminus B_{1/2})} r^{\sqrt{N-1}+1} \left(r^{2-N} \int_{1/2}^{1} \frac{1}{t^{N-1}}\, dt\right)^{1/2},
\end{aligned}$$

and (2.3) is proved. □

Proof of Theorem 1.6. Let $0 < r \leqslant 1$. Then, there exist $m \in \mathbb{N}$ and $1/2 < r_1 \leqslant 1$ such that $r = r_1/2^{m-1}$. Since u is radial we have $u(r_1) \leqslant \|u\|_{L^\infty(B_1 \setminus B_{1/2})} \leqslant \gamma_N \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})}$, where γ_N depends only on N. From this and Proposition 2.2, it follows that

$$\begin{aligned}
\left|u(r)\right| &\leqslant \left|u(r_1) - u(r)\right| + \left|u(r_1)\right| \leqslant \sum_{i=1}^{m-1} \left|u\left(\frac{r_1}{2^{i-1}}\right) - u\left(\frac{r_1}{2^i}\right)\right| + \left|u(r_1)\right| \\
&\leqslant K'_N \|\nabla u\|_{L^2(B_1 \setminus B_{1/2})} \sum_{i=1}^{m-1} \left(\frac{r_1}{2^{i-1}}\right)^{-N/2+\sqrt{N-1}+2} + \gamma_N \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})} \\
&\leqslant \left(K'_N \sum_{i=1}^{m-1} \left(\frac{r_1}{2^{i-1}}\right)^{-N/2+\sqrt{N-1}+2} + \gamma_N\right) \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})}.
\end{aligned} \tag{2.4}$$

- If $2 \leqslant N < 10$, we have $-N/2 + \sqrt{N-1} + 2 > 0$. Then

$$\sum_{i=1}^{m-1} \left(\frac{r_1}{2^{i-1}}\right)^{-N/2+\sqrt{N-1}+2} \leqslant \sum_{i=1}^{\infty} \left(\frac{1}{2^{i-1}}\right)^{-N/2+\sqrt{N-1}+2},$$

  which is a convergent series. Applying (2.4), statement (i) of the theorem is proved.
- If $N = 10$, we have $-N/2 + \sqrt{N-1} + 2 = 0$. From (2.4) we obtain

$$\left|u(r)\right| \leqslant \left(K'_N(m-1) + \gamma_N\right) \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})}$$

$$= \left(K'_N \left(\frac{\log r_1 - \log r}{\log 2} \right) + \gamma_N \right) \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})}$$

$$\leqslant \left(\frac{K'_N}{\log 2} + \gamma_N \right) \big(|\log r| + 1 \big) \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})},$$

which gives statement (ii).

- If $N > 10$, we have $-N/2 + \sqrt{N-1} + 2 < 0$. Then

$$\sum_{i=1}^{m-1} \left(\frac{r_1}{2^{i-1}} \right)^{-N/2+\sqrt{N-1}+2} = \frac{r^{-N/2+\sqrt{N-1}+2} - r_1^{-N/2+\sqrt{N-1}+2}}{(1/2)^{-N/2+\sqrt{N-1}+2} - 1}.$$

From this and (2.4), we conclude

$$\big|u(r)\big| \leqslant \left(\frac{K'_N}{(1/2)^{-N/2+\sqrt{N-1}+2} - 1} + \gamma_N \right) r^{-N/2+\sqrt{N-1}+2} \|u\|_{H^1(B_1 \setminus \overline{B_{1/2}})},$$

which completes the proof. □

Proof of Theorem 1.7.

(i) We first observe that $(-r^{N-1} u_r)' = r^{N-1} g(u) \geqslant 0$. Hence $-r^{N-1} u_r$ is a positive nondecreasing function and so is $r^{2N-2} u_r^2$. Thus, for $0 < r \leqslant 1/2$, we have

$$\int_0^{2r} t^{N-1} u_r(t)^2 \, dt \geqslant \int_r^{2r} t^{N-1} u_r(t)^2 \, dt = \int_r^{2r} t^{2N-2} u_r(t)^2 \frac{1}{t^{N-1}} \, dt$$

$$\geqslant r^{2N-2} u_r(r)^2 \int_r^{2r} \frac{1}{t^{N-1}} \, dt = r^{2N-2} u_r(r)^2 r^{2-N} \int_1^2 \frac{1}{t^{N-1}} \, dt.$$

From this and Lemma 2.1 we obtain (i).

(ii) Consider the function $\Psi(r) = -N r^{1-1/N} u_r(r^{1/N})$, $r \in (0,1]$. It is easy to check that $\Psi'(r) = g(u(r^{1/N}))$, $r \in (0,1]$. As g is nonnegative and nondecreasing we have that Ψ is a nonnegative nondecreasing concave function. It follows immediately that $0 \leqslant \Psi'(r) \leqslant \Psi(r)/r$, $r \in (0,1]$; which becomes

$$0 \leqslant -(N-1) r^{-1/N} u_r\big(r^{1/N}\big) - u_{rr}\big(r^{1/N}\big) \leqslant -N r^{-1/N} u_r\big(r^{1/N}\big), \quad r \in (0,1].$$

Hence

$$r^{-1/N} u_r\big(r^{1/N}\big) \leqslant u_{rr}\big(r^{1/N}\big) \leqslant -(N-1) r^{-1/N} u_r\big(r^{1/N}\big), \quad r \in (0,1].$$

Therefore $|u_{rr}(r)| \leqslant (N-1)|u_r(r)|/r$, $r \in (0,1]$; and (ii) follows from (i).

(iii) An easy computation shows that

$$u_{rrr} = -u_r g'(u) - \frac{N-1}{r}u_{rr} + \frac{N-1}{r^2}u_r, \quad r \in (0,1].$$

On the other hand, it is proved in [5, Th. 1.8 (c)] that $g'(u(r)) \leqslant h_N/r^2$, $r \in (0,1]$, for some constant h_N. Since we have shown $|u_{rr}(r)| \leqslant (N-1)|u_r(r)|/r$, $r \in (0,1]$ in the proof of statement (ii), it follows from the above formula $|u_{rrr}(r)| \leqslant s_N|u_r(r)|/r^2$, $r \in (0,1]$, for some constant s_N depending only on N. Recalling (i), the proof is now completed. □

To deduce Theorem 1.2 from Theorems 1.6 and 1.7 we need the following lemma.

Lemma 2.3. *Let $N \geqslant 2$, $g \in C^1(\mathbb{R})$ nonnegative and nondecreasing function and u a radially decreasing solution of* (1.1) (*neither $u \in H^1(B_1)$ nor u is semi-stable is required*). *Then*

(i) $r^{N-1}|u_r|$ *is nondecreasing for $r \in (0,1]$.*
(ii) $r^{-1}|u_r|$ *is nonincreasing for $r \in (0,1]$.*
(iii) $\max_{t\in[1/2,1]} |u_r(t)| \leqslant 2^{N-1} \min_{t\in[1/2,1]} |u_r(t)|$.
(iv) $\|\nabla u\|_{L^2(B_1\setminus B_{1/2})} \leqslant q_N \min_{t\in[1/2,1]} |u_r(t)|$, *for a certain constant q_N depending only on N.*

Proof.

(i) Since $u_r < 0$ we have $(r^{N-1}|u_r|)' = r^{N-1}g(u) \geqslant 0$.
(ii) As in the proof of statement (ii) of Theorem 1.7 we have that the function $\Psi(r) = -Nr^{1-1/N}u_r(r^{1/N})$ is nonnegative, nondecreasing and concave for $r \in (0,1]$. Therefore $\Psi(r)/r = -Nr^{-1/N}u_r(r^{1/N})$ is nonincreasing, and (ii) follows immediately.
(iii) Take $r_1, r_2 \in [1/2,1]$ such that $|u_r(r_1)| = \min_{t\in[1/2,1]} |u_r(t)|$ and $|u_r(r_2)| = \max_{t\in[1/2,1]} |u_r(t)|$.
If $r_2 \leqslant r_1$, we deduce from (i) that $|u_r(r_2)| \leqslant (r_1/r_2)^{N-1}|u_r(r_1)| \leqslant 2^{N-1}|u_r(r_1)|$.
If $r_2 > r_1$, we deduce from (ii) that $|u_r(r_2)| \leqslant (r_2/r_1)|u_r(r_1)| \leqslant 2|u_r(r_1)| \leqslant 2^{N-1}|u_r(r_1)|$.
(iv) We see at once that

$$\|\nabla u\|_{L^2(B_1\setminus B_{1/2})} \leqslant \big(\text{measure}(B_1 \setminus B_{1/2})\big)^{1/2} \max_{t\in[1/2,1]} \big|u_r(t)\big|,$$

and (iv) follows from (iii). □

Proof of Theorem 1.2. In the case $N = 1$, it is well known that $u \in C^3$. Taking into account the signs of u^*, u^*_r, u^*_{rr} and u^*_{rrr}, it is easy to check that $u^*(r) \leqslant |u^*_r(1)|(1-r) \leqslant 2|u^*_r(1/2)|(1-r)$, $0 \leqslant r \leqslant 1$, which is the desired conclusion wit $C_1 = 2$. Hence, for the rest of the proof we will suppose $N \geqslant 2$.

As we have mentioned, it is well known that u^* is a semi-stable radially decreasing $H_0^1(B_1)$ solution of (1.1) for $g(s) = \lambda^* f(s)$. Hence, we can apply to u^* the results obtained in Theorems 1.6 and 1.7 and Lemma 2.3.

Let us first prove (i), (ii) and (iii) for $r \in (0,1/2)$. Since $u^*(1) = 0$, and on account of statement (iv) of Lemma 2.3, we have $\|u^*\|_{H^1(B_1\setminus\overline{B_{1/2}})} \leqslant h_N\|\nabla u^*\|_{L^2(B_1\setminus B_{1/2})} \leqslant h'_N \min_{t\in[1/2,1]} |u^*_r(t)|$, for certain constants h_N, h'_N depending only on N. From this and Theorem 1.6:

(i) follows from the inequality $1 \leqslant 2(1-r)$, for $r \in (0, 1/2)$.
(ii) follows from the inequality $|\log r| + 1 \leqslant \frac{\log 2+1}{\log 2}|\log r|$, for $r \in (0, 1/2)$.
(iii) follows from the inequality

$$r^{-N/2+\sqrt{N-1}+2} \leqslant \frac{(1/2)^{-N/2+\sqrt{N-1}+2}}{(1/2)^{-N/2+\sqrt{N-1}+2} - 1}\left(r^{-N/2+\sqrt{N-1}+2} - 1\right), \quad \text{for } r \in (0, 1/2).$$

We next show (i), (ii) and (iii) for $r \in [1/2, 1]$. From statement (iii) of Lemma 2.3 it follows that

$$u^*(r) = \int_r^1 \left|u_r^*(t)\right| dt \leqslant (1-r)2^{N-1} \min_{t\in[1/2,1]} \left|u_r^*(t)\right|, \quad \forall r \in [1/2, 1],$$

which is the desired conclusion if $N < 10$. If $N = 10$, our claim follows from the inequality $1 - r \leqslant |\log r|$, for $r \in [1/2, 1]$. Finally, if $N > 10$, the desired conclusion follows immediately from the inequality $1 - r \leqslant z_N(r^{-N/2+\sqrt{N-1}+2} - 1)$, for $r \in [1/2, 1]$, for a certain constant z_N.

We now prove statement (iv). In the case $k = 1$ and $r \in (0, 1/2)$, it follows immediately from statement (i) of Theorem 1.7 and statement (iv) of Lemma 2.3. The case $k = 1$ and $r \in [1/2, 1]$ is also obvious on account of statement (iii) of Lemma 2.3 and the inequality $1 \leqslant r^{-N/2+\sqrt{N-1}+1}$, for $r \in [1/2, 1]$, for $N \geqslant 10$.

Finally, as in the proof of statement (ii) and (iii) of Theorem 1.7, we have $|u_{rr}^*(r)| \leqslant (N-1)|u_r^*(r)|/r$ and $|u_{rrr}^*(r)| \leqslant s_N|u_r^*(r)|/r^2$, for $r \in (0, 1]$, which gives statement (iv) for $k = 2, 3$ from the case $k = 1$. □

3. A family of semi-stable solutions

Theorem 3.1. *Let $h \in (C^2 \cap L^1)(0, 1]$ be a nonnegative function and consider*

$$\Phi(r) = r^{2\sqrt{N-1}}\left(1 + \int_0^r h(s)\, ds\right) \quad \forall r \in (0, 1].$$

Define $u_r < 0$ by

$$\Phi'(r) = (N-1)r^{N-3}u_r(r)^2 \quad \forall r \in (0, 1].$$

Then, for $N \geqslant 10$, u is a semi-stable radially decreasing unbounded $H^1(B_1)$ solution of a problem of the type (1.1)*, where u is any function with radial derivative u_r.*

To prove Theorem 3.1 we need the following lemma, which is a generalization of the classical Hardy inequality:

Lemma 3.2. *Let $\Phi \in C^1(0, L)$, $0 < L \leqslant \infty$, satisfying $\Phi' > 0$. Then*

$$\int_0^L \frac{4\Phi^2}{\Phi'} \xi'^2 \geqslant \int_0^L \Phi' \xi^2,$$

for every $\xi \in C^\infty(0, L)$ with compact support.

Proof. Integrating by parts and applying Cauchy–Schwarz we obtain

$$\int_0^L \Phi'\xi^2 = -2\int_0^L \Phi\xi\xi' \leqslant 2\int_0^L \frac{|\Phi|}{\sqrt{\Phi'}} |\xi'| \sqrt{\Phi'}|\xi| \leqslant 2\left(\int_0^L \frac{\Phi^2}{\Phi'}\xi'^2\right)^{1/2} \left(\int_0^L \Phi'\xi^2\right)^{1/2},$$

which establishes the desired inequality. □

In the case $\Phi(r) = ((N-2)/4)r^{N-2}$, $r > 0$, the above lemma is the Hardy inequality for radial functions in $\mathbb{R}^N$, $N > 2$.

Proof of Theorem 3.1. First of all, since $\Phi \in C^1(0, 1] \cap C[0, 1]$ is an increasing function, we obtain $\Phi' \in L^1(0, 1)$ and hence $r^{N-1}u_r^2 = r^2\Phi'/(N-1) \in L^1(0, 1)$, which gives $u \in H^1(B_1)$.

On the other hand, since $\Phi'(r) \geqslant 2\sqrt{N-1}r^{2\sqrt{N-1}-1}$, $r \in (0, 1]$, we deduce $|u_r(r)| \geqslant \sqrt{2}(N-1)^{-1/4}r^{-N/2+\sqrt{N-1}+1}$, $r \in (0, 1]$. As $N \geqslant 10$, we have $-N/2 + \sqrt{N-1} + 1 \leqslant -1$. It follows that $u_r \notin L^1(0, 1)$ and, since u is radially decreasing, we obtain $\lim_{r\to 0} u(r) = +\infty$.

Since $h \in C^2(0, 1]$, it follows that $u_r \in C^2(0, 1]$. Therefore, $\Delta u \in C^1(\overline{B_1} \setminus \{0\})$. Hence, taking $g \in C^1(\mathbb{R})$ such that $g(s) = -\Delta u(u^{-1}(s))$, for $s \in [u(1), +\infty)$, we conclude that u is solution of a problem of the type (1.1).

It remains to prove that u is semi-stable. Taking into account that $u_r \neq 0$ in $(0, 1]$ and applying [5, Lem. 2.1], the semi-stability of u is equivalent to

$$\int_0^1 r^{N-1}u_r^2\xi'^2\,dr \geqslant (N-1)\int_0^1 r^{N-3}u_r^2\xi^2\,dr, \tag{3.1}$$

for every $\xi \in C^\infty(0, 1)$ with compact support.

For this purpose, we will apply the lemma above. From the definition of Φ it is easily seen that $\Phi' \geqslant 2\sqrt{N-1}\Phi/r$, $r \in (0, 1]$. It follows that

$$\frac{\Phi' r^2}{N-1} \geqslant \frac{4\Phi^2}{\Phi'} \quad \text{in } (0, 1].$$

Finally, since $\Phi' r^2/(N-1) = r^{N-1}u_r^2$ and $\Phi' = (N-1)r^{N-3}u_r^2$ in $(0, 1]$, we deduce (3.1) by applying Lemma 3.2. □

As an application of Theorem 3.1 we have the following results, which show the impossibility of obtaining any pointwise estimate for $|u_r|$, $|u_{rr}|$ or $|u_{rrr}|$ if the positivity of g, g' or g'' is not satisfied, for semi-stable radially decreasing $H^1(B_1)$ solutions of a problem of the type (1.1) and $N \geqslant 10$.

Proposition 3.3. *Let $\{r_n\} \subset (0,1]$, $\{M_n\} \subset \mathbb{R}^+$ two sequences with $r_n \downarrow 0$. Then, for $N \geqslant 10$, there exists $u \in H^1(B_1)$, which is a semi-stable radially decreasing unbounded solution of a problem of the type* (1.1)*, satisfying*

$$\left|u_r(r_n)\right| \geqslant M_n \quad \forall n \in \mathbb{N}.$$

Proof. It is easily seen that for every sequences $\{r_n\} \subset (0,1]$, $\{y_n\} \subset \mathbb{R}^+$, with $r_n \downarrow 0$, there exists a nonnegative function $h \in (C^2 \cap L^1)(0,1]$ satisfying $h(r_n) = y_n$. Take $y_n = (N-1) \times M_n^2 r_n^{N-2\sqrt{N-1}-3}$ and apply Theorem 3.1 with this function h. It is clear, from the definition of Φ, that $\Phi'(r) \geqslant h(r) r^{2\sqrt{N-1}}$, $r \in (0,1]$. Hence

$$(N-1) r_n^{N-3} u_r(r_n)^2 = \Phi'(r_n) \geqslant h(r_n) r_n^{2\sqrt{N-1}} = y_n r_n^{2\sqrt{N-1}} = (N-1) r_n^{N-3} M_n^2,$$

and the proposition follows. □

Corollary 3.4. *Let $N \geqslant 10$. There does not exist a function $\psi : (0,1] \to \mathbb{R}^+$ with the following property: for every $u \in H^1(B_1)$ semi-stable radially decreasing solution of a problem of the type* (1.1)*, there exist $C > 0$ and $\varepsilon \in (0,1]$ such that $|u_r(r)| \leqslant C\psi(r)$ for $r \in (0,\varepsilon]$.*

Proof. Suppose that such a function ψ exists and consider the sequences $r_n = 1/n$, $M_n = n\psi(1/n)$. By the proposition above, there exists $u \in H^1(B_1)$, which is a semi-stable radially decreasing unbounded solution of a problem of the type (1.1), satisfying $|u_r(1/n)| \geqslant n\psi(1/n)$, a contradiction. □

Proposition 3.5. *Let $\{r_n\} \subset (0,1]$, $\{M_n\} \subset \mathbb{R}^+$ two sequences with $r_n \downarrow 0$. Then, for $N \geqslant 10$, there exists $u \in H^1(B_1)$, which is a semi-stable radially decreasing unbounded solution of a problem of the type* (1.1) *with $g \geqslant 0$, satisfying*

$$\left|u_{rr}(r_n)\right| \geqslant M_n \quad \forall n \in \mathbb{N}.$$

Proof. Let $h \in C^2(0,1]$, increasing, satisfying $0 \leqslant h \leqslant 1$. Define Φ and u_r as in Theorem 3.1. We claim that

(i) u is a semi-stable radially decreasing unbounded $H^1(B_1)$ solution of a problem of the type (1.1) with $g \geqslant 0$.
(ii) $|u_r| \leqslant D_N r^{-N/2+\sqrt{N-1}+1}$, $\forall r \in (0,1]$, where D_N only depends on N.
(iii) $-u_{rr} \geqslant E_N h'(r) r^{-N/2+\sqrt{N-1}+2} - F_N r^{-N/2+\sqrt{N-1}}$, $\forall r \in (0,1]$, where $E_N > 0$ and F_N only depend on N.

Since h is positive and increasing, then $\Phi'' > 0$. Hence $(N-1) r^{N-3} u_r^2$ is increasing and so is $r^{2N-2} u_r^2$. This implies that $-r^{N-1} u_r$ is increasing, which is equivalent to the positiveness of g.

On the other hand note that, since $0 \leqslant h \leqslant 1$, we obtain $\Phi'(r) \leqslant G_N r^{2\sqrt{N-1}-1}$ in $(0,1]$, for a constant G_N. Hence, from the definition of u_r we obtain (ii).

To prove (iii) observe that, from the positiveness of h, we obtain $\Phi''(r) \geqslant r^{2\sqrt{N-1}}h'(r)$ in $(0,1]$. On the other hand, from the definition of u_r we have $\Phi''(r) = (N-1)((N-3)r^{N-4}u_r^2 + 2u_r u_{rr} r^{N-3})$. Therefore, by (ii) and the previous inequality we obtain (iii).

Finally, it is easily seen that for every sequences $\{r_n\} \subset (0,1]$, $\{y_n\} \subset \mathbb{R}^+$, with $r_n \downarrow 0$, there exists $h \in C^2(0,1]$, increasing, satisfying $0 \leqslant h \leqslant 1$ and $h'(r_n) = y_n$. Take y_n such that $E_N y_n r_n^{-N/2+\sqrt{N-1}+2} - F_N r_n^{-N/2+\sqrt{N-1}} = M_n$. Applying (iii) we deduce $-u_{rr}(r_n) \geqslant M_n$ and the proof is complete. □

Corollary 3.6. *Let $N \geqslant 10$. There does not exist a function $\psi : (0,1] \to \mathbb{R}^+$ with the following property: for every $u \in H^1(B_1)$ semi-stable radially decreasing solution of a problem of the type* (1.1) *with $g \geqslant 0$, there exist $C > 0$ and $\varepsilon \in (0,1]$ such that $|u_{rr}(r)| \leqslant C\psi(r)$ for $r \in (0,\varepsilon]$.*

Proof. Arguing as in Corollary 3.4 and using Proposition 3.5, we conclude the proof of the corollary. □

Proposition 3.7. *Let $\{r_n\} \subset (0,1]$, $\{M_n\} \subset \mathbb{R}^+$ two sequences with $r_n \downarrow 0$. Then, for $N \geqslant 10$, there exists $u \in H^1(B_1)$, which is a semi-stable radially decreasing unbounded solution of a problem of the type* (1.1) *with $g, g' \geqslant 0$, satisfying*

$$\left|u_{rrr}(r_n)\right| \geqslant M_n \quad \forall n \in \mathbb{N}.$$

Lemma 3.8. *For any dimension $N \geqslant 10$, there exists $\varepsilon_N > 0$ with the following property: for every $h \in C^2(0,1] \cap C^1[0,1]$ satisfying $h(0) = 0$, $0 \leqslant h' \leqslant \varepsilon_N$ and $h'' \leqslant 0$, u is a semi-stable radially decreasing unbounded $H^1(B_1)$ solution of a problem of the type* (1.1) *with $g, g' \geqslant 0$, where u_r is defined in terms of h as in Theorem* 3.1.

Proof. Similarly as in the proof of Proposition 3.5 (item (i)), $h' \geqslant 0$ implies that u is a semi-stable radially decreasing unbounded $H^1(B_1)$ solution of a problem of the type (1.1) with $g \geqslant 0$.

On the other hand, from the definition of Φ and u_r it follows easily that

$$\begin{aligned} u_r &= -\sqrt{(N-1)^{-1} r^{3-N} \Phi'} \\ &= -r^{-N/2+\sqrt{N-1}+1} \sqrt{2(N-1)^{-1/2}\left(1 + \int_0^r h\right) + (N-1)^{-1} r h}. \end{aligned}$$

Put this last expression in the form $u_r = -r^{-N/2+\sqrt{N-1}+1}\varphi(r)$, where $\varphi(r)$ (and of course u_r) depends on h. Now consider the set $X = \{h \in C^2(0,1] \cap C^1[0,1]\colon\ h(0) = 0,\ 0 \leqslant h',\ h'' \leqslant 0\}$ and the norm $\|h\|_X = \|h'\|_{L^\infty(0,1)}$. Taking $\|h\|_X \to 0$, we have

$$\begin{gathered} \lim_{\|h\|_X \to 0} \varphi = \sqrt{2}(N-1)^{-1/4}, \qquad \lim_{\|h\|_X \to 0} \varphi' = 0, \\ \lim_{\|h\|_X \to 0} \left(\varphi'' - \frac{(N-1)^{-1} r h''}{2\varphi}\right) = 0, \end{gathered} \tag{3.2}$$

where all the limits are taken uniformly in $r \in (0,1]$. On the other hand, it is easy to check that

$$r^2 g'(u) = \frac{-r^2 u_{rrr}}{u_r} - \frac{(N-1) r u_{rr}}{u_r} + (N-1)$$
$$= \frac{-r^2 \varphi''}{\varphi} - \frac{(2\sqrt{N-1}+1) r \varphi'}{\varphi} + \frac{(N-2)^2}{4}.$$

Hence, from (3.2), we can assert that, for $h \in X$ with small $\|h\|_X$, $r^2 g'(u) > 0$ in $(0, 1]$, and the lemma follows. □

Proof of Proposition 3.7. We follow the notation used in the previous lemma. From (3.2), we deduce that

$$\lim_{\|h\|_X \to 0} \left(r^{N/2-\sqrt{N-1}+1} u_{rrr} + \frac{(N-1)^{-1} r^3 h''}{2\varphi} \right) = \sigma,$$

uniformly in $r \in (0, 1]$, where $\sigma = -(-N/2 + \sqrt{N-1} + 1)(-N/2 + \sqrt{N-1})\sqrt{2} \times (N-1)^{-1/4} < 0$. Then, taking $\varepsilon'_N > 0$ sufficient small (possibly less than ε_N), we have that

$$r^{N/2-\sqrt{N-1}+1} u_{rrr} \geqslant -\left(\frac{(N-1)^{-1} r^3 h''}{2\sqrt{2}(N-1)^{-1/4}+1} \right) + \sigma - 1, \quad \forall r \in (0, 1],$$

for $\|h\|_X \leqslant \varepsilon'_N$.

Finally, it is easily seen that for every sequences $\{r_n\} \subset (0, 1]$, $\{y_n\} \subset \mathbb{R}^+$, with $r_n \downarrow 0$, there exists $h \in X$, with $\|h\|_X \leqslant \varepsilon'_N$, satisfying $h''(r_n) = -y_n$. (Take, for instance $h(r) = \int_0^r z(t)\,dt$, where $z \in C^1(0, 1] \cap C[0, 1]$ is decreasing, $0 \leqslant z(t) \leqslant \varepsilon'_N$ and satisfies $z'(r_n) = -y_n$.) Take y_n such that $r_n^{N/2-\sqrt{N-1}+1} M_n = (\frac{(N-1)^{-1} r_n^3 y_n}{2\sqrt{2}(N-1)^{-1/4}+1}) + \sigma - 1$. Applying the above inequality, we obtain $u_{rrr}(r_n) \geqslant M_n$ and the proof is complete. □

Corollary 3.9. *Let $N \geqslant 10$. There does not exist a function $\psi : (0, 1] \to \mathbb{R}^+$ with the following property: for every $u \in H^1(B_1)$ semi-stable radially decreasing solution of a problem of the type* (1.1) *with $g, g' \geqslant 0$, there exist $C > 0$ and $\varepsilon \in (0, 1]$ such that $|u_{rrr}(r)| \leqslant C\psi(r)$ for $r \in (0, \varepsilon]$.*

Proof. Applying Proposition 3.7, this follows by the same method as in Corollaries 3.4 and 3.6. □

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References

[1] H. Brezis, T. Cazenave, Y. Martel, A. Ramiandrisoa, Blow up for $u_t - \Delta u = g(u)$ revisited, Adv. Differential Equations 1 (1996) 73–90.

[2] H. Brezis, J.L. Vázquez, Blow-up solutions of some nonlinear elliptic problems, Rev. Mat. Univ. Complut. Madrid 10 (1997) 443–469.

[3] X. Cabré, Regularity of minimizers of semilinear elliptic problems up to dimension 4, Comm. Pure Appl. Math. 63 (2010) 1362–1380.

[4] X. Cabré, A. Capella, On the stability of radial solutions of semilinear elliptic equations in all of $\mathbb{R}^n$, C. R. Math. Acad. Sci. Paris 338 (2004) 769–774.
[5] X. Cabré, A. Capella, Regularity of radial minimizers and extremal solutions of semilinear elliptic equations, J. Funct. Anal. 238 (2006) 709–733.
[6] X. Cabré, A. Capella, M. Sanchón, Regularity of radial minimizers of reaction equations involving the p-Laplacian, Calc. Var. Partial Differential Equations 34 (2009) 475–494.
[7] M.G. Crandall, P.H. Rabinowitz, Some continuation and variational methods for positive solutions of nonlinear elliptic eigenvalue problems, Arch. Ration. Mech. Anal. 58 (1975) 207–218.
[8] E. Giusti, Minimal Surfaces and Functions of Bounded Variation, Monogr. Math., vol. 80, Birkhäuser Verlag, 1984.
[9] D.D. Joseph, T.S. Lundgren, Quasilinear Dirichlet problems driven by positive sources, Arch. Ration. Mech. Anal. 49 (1973) 241–269.
[10] F. Mignot, J.-P. Puel, Sur une classe de problèmes non linéaires avec non linéairité positive, croissante, convexe, Comm. Partial Differential Equations 5 (1980) 791–836.
[11] G. Nedev, Regularity of the extremal solution of semilinear elliptic equations, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 997–1002.
[12] S. Villegas, Asymptotic behavior of stable radial solutions of semilinear elliptic equations in $\mathbb{R}^N$, J. Math. Pures Appl. 88 (2007) 241–250.